Nonlinear controllability of singularly perturbed models of power flow networks

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Abstract—A method based on differential geometric control theory is presented intended to provide insight into how the nodes of a power network can affect each other. In this preliminary report, we consider a simple model of a power system derived from singular perturbation of the power flow equations. It is shown that such a model is accessible, and that for simple chain topology the network is actually feedback linearizable. The result is illustrated numerically. This simple example is a precursor for more interesting models of networks.

I. INTRODUCTION

Dynamical analysis of large electric power networks is becoming increasingly important as power systems have become larger and more interconnected and are operated close to stability limits. Power networks are high dimensional dynamical systems composed of heterogeneous components governed by nonlinear evolution equations and subject to disturbances. These facts make them difficult to analyze.

An important aspect of the dynamical behavior of power systems that is particularly difficult to codify is that directly related to its network structure. The connectivity implies that any change at one bus necessarily affects all buses, so that fundamental questions such as how power from a given bus is distributed in the network are quite subtle to deal with.

Power networks can be represented as planar graphs, which are much-studied objects, however being neither regular nor random they fall into a category that is somewhat resistant to conventional approaches. Power grids are typically modeled mesoscopically, meaning that they generally consist of tens to thousands of nodes, and their dynamics take place in a state space with two to perhaps twenty dimensions per node. See below for a more detailed description of dynamical power grid models. In this paper we use geometric control theory to analyze the network-dependent structure in power systems. This paper begins the investigation by considering perhaps the simplest dynamics possible based on conventional power flow equations. See [2] for another application of control methods to power systems.

We construct minimally complicated dynamical models of power networks as affine nonlinear control systems and use these to investigate how the inputs of a given node of the network influence the other nodes. This question is of enormous importance for the study of stability, and also might have direct economic implications in regard to how a given generator-load pair interact.

II. DYNAMICAL MODELS OF POWER NETWORKS

An electric power network can be usefully modeled in the context of what are known as “coupled cell” systems in the nonlinear dynamics and control literature. The cells correspond to busses of the power networks, and devices connected to the busses such as generators, loads, voltage control devices or other things define internal states and dynamics of the cells. A characteristic of power networks that simplifies this analysis is when the coupling between nodes occurs solely through the power flow equations. This is not always strictly true as certain devices such as transformers and static VAR compensators are located between busses, but these devices can often be absorbed into redefinitions of effective busses so that the coupled cell networks correspond to partial networks of a specified voltage levels. In this case the evolution of the internal states of the devices sees only the voltage phasor of the bus to which it is connected, and also that the evolution of the voltage phasor depends only on the bus’s own internal states and the voltages of the other busses to which it is connected (but not the internal states of those busses).

The nature of the evolutionary equations depend on the modeling paradigm. The simplest representation of an operating power grid is its power flow solution. In this situation real and reactive power generation and consumption is regarded as constant, and the solution is a set of constant voltage phasors for the nodes. In this context there is no dynamics at all. Adding internal node dynamics to this picture results in a differential-algebraic system of equations (DAE) in which case the equations for the \( j \)th node of the bus take the form

\[
\dot{z}_j = f_j(z_j, V_j) \tag{II.1}
\]

\[
0 = P_j + iQ_j - \Sigma_k V_j V_{jk} \tag{II.2}
\]

where \( z_j \) is the vector of internal states, \( V_j \) is the voltage phasor, \( P_j \) and \( Q_j \) are the real and reactive power injection at the node which can be positive or negative and may depend on internal states and the voltage phasor. \( Y_{jk} \) is the...
admittance of the line connecting node \( j \) to node \( k \), which vanishes if the two nodes are not directly connected.

The assumption of exact solution of the power flow equations for all times is an approximation that amounts to regarding the dynamics at the bus as infinitely fast. This can be a convenient assumption, but is not really justified from first principles. In fact, there will be nontrivial dynamics on fast, but finite, time scales that depends in a complicated way on the specifics of the devices connected to the bus. The infinitely fast dynamics of the power flow equations is often justified by arguing that no realistic modeling of the fast dynamics in general is possible. While this may be so, there is another solution seen in the literature \([6]\), \([3]\) that has its own advantages, which is to replace II.2 with a singularly perturbed version (SPDE) that reflects that the voltage phasor will exhibit fast dynamics. This type of approximation cannot be used thoughtlessly (see, e.g., chapter 8 of \([3]\)), but sufficiently close to a hyperbolic fixed point there is reason to believe that smooth fast dynamics exist, and a perturbation of this type can be very useful.

The precise way in which the time evolution of the phasor components is associated with the mismatch of power flow is not uniquely determined. However, in some common situations \([3]\) the lore of the electrical engineering literature attributes evolution of the phasor angle to the real power mismatch, and that of the voltage amplitude to the reactive power mismatch. In this case, the (fully dynamical) equations for the \( j \)th node become

\[
\begin{align*}
\dot{z}_j &= f_j(z_j, V_j) \quad \text{(II.3)} \\
\epsilon \dot{\theta}_j &= P_j - \text{Re}(\Sigma_k V_j V_k^* Y_{jk}) \quad \text{(II.4)} \\
\epsilon \dot{V}_j &= Q_j - \text{Im}(\Sigma_k V_j V_k^* Y_{jk}) \quad \text{(II.5)}
\end{align*}
\]

where \( \epsilon \ll 1 \) is a representation of the separation of fast and slow time scales.

The equations II.3-II.5 have many advantages over II.1-II.2 from the point of view of analysis and simulation. Simulations of DAEs can be hampered by the need to find good initial conditions which also affects the ability of such simulations to deal with protection events such as load shedding or line and generator tripping. Also, being a smooth dynamical system the tools of nonlinear dynamics such as bifurcation theory can be applied more easily. Finally, as will be seen below, the methods of differential geometric control theory can be easily adapted to provide a means of looking at the node influence problem that is the heart of this paper.

It should be acknowledged that the nonuniqueness of the association of the evolution of states to the power mismatch is somewhat problematic from a modeling point of view. Mathematically, however, one can think in terms of “blowups” of singular problems in the sense of Takens \([8]\). This theory guarantees that when the techniques are valid (ie, away from singularities of the DAE \([3]\)) and properly interpreted any such desingularization can be used to provide information about the associated DAE. Experience with these models seems to support these ideas, and we take the position that any stable blowup of the DAE is an equally good representative of the fast dynamics as far as the large scale, long time structure of the network is concerned. Moreover, in the differential geometric context to be laid out below, many considerations are invariant under diffeomorphism, so that a wide class of possible desingularizations are equivalent. In fact, the controllability results we obtain are in a way independent of this choice, though the details of the dynamical trajectories certainly depend on it.

### III. Affine Control Model of Power Flow Networks

For the simplest investigation of the influence of the nodes through the network, we will ignore the presence of the internal states. It can be easily seen that the nature of the coupled cell structure and the power flow coupling ensures that questions of influence of internal states or parameters will ultimately depend on the present analysis. For computational and analytic reasons we adopt rectangular coordinates for the complex phasors \( V_j = x_j + iy_j \). In this case the power flow equations depend only on two types of quadratic polynomials. For notational simplicity, we define

\[
D_{jk} = x_j x_k + y_j y_k \\
C_{jk} = x_j y_k - x_k y_j
\]

and the equations we consider are

\[
\begin{align*}
\dot{x}_j &= P_j + \Sigma_k (D_{jk} G_{jk} - C_{jk} B_{jk}) \quad \text{(III.6)} \\
\dot{y}_j &= Q_j - \Sigma_k (D_{jk} B_{jk} + C_{jk} G_{jk}) \quad \text{(III.7)}
\end{align*}
\]

where \( G_{jk} \) is the real part of \( Y_{jk} \) (the conductance) and \( B_{jk} \) is the imaginary part (the susceptance). We have absorbed the singular parameter \( \epsilon \) into the definition of the unit of time which is permissible at this stage since we are assuming no internal (slow time) dynamics are present. In this context the parameters \( P_j \) and \( Q_j \) are constants, and will be selectively identified as control inputs for the node whose influence we wish to investigate.

To construct the affine control system, we assume that we have control of the inputs at some bus, which we label as node 1. That is, we identify \( P_1 \) and \( Q_1 \) as control inputs. Denoting by \( x \) the state vector constructed by concatenating all \( n = 2N \) phasor components, where \( N \) is the number of busses, the resulting control system is

\[
x = f(x) + g_1 u_P + g_2 u_Q \quad \text{(III.8)}
\]

where \( g_1 = (1, 0, 0, \ldots, 0)^T \) and \( g_2 = (0, 1, 0, \ldots, 0)^T \) are the control vector fields associated with the inputs \( u_P \) and \( u_Q \) which are the departures of the power inputs from their equilibrium values. \( f(x) \) is the rest of III.6 and III.7 after the control vectors have been isolated and is referred to as the drift of the affine control system.

Putting the system into this form allows us to address the question of how manipulating the inputs to node 1 can affect
the state of the system. We use the apparatus of differential geometric control. [5], [4], [9] The problem we consider is a type of input-output control. Specifically, we wish to find control laws for the power inputs at bus 1 that will steer the voltage phasor at another bus to prescribed values. We will achieve this by the technique of input-output linearization.

As the starting point of the analysis we consider the simple example of a network with chain topology of length $N$ and consider manipulating the inputs at the endpoint. See Figure 1. It will be seen that this network is particularly well suited to these methods, and that for arbitrary networks, the input-output goal is achieved by identifying the chain as the shortest path in a connected network that links the control bus to the output bus. This will be discussed further below.

IV. ACCESSIBILITY PROPERTIES

To begin, we define the fundamental object of study:

Definition 1: The node accessibility algebra of node 1 is the smallest involutive, drift invariant distribution that contains the control vector fields $g_1$ and $g_2$. Note that this distribution comprises a Lie algebra.

A fundamental result from differential geometric control theory is that if the dimension of this distribution is $d$, then the system III.8 can be diffeomorphically put into triangular form in a neighborhood of a regular point $x$ such that an $n - d$ dimensional subspace is uncontrolled, and the reachable set of the control system contains an open subset of the complementary $d$ dimensional subspace. The corollary is that when the distribution has full rank, the system’s reachable set is an open subset of the state space. This is a necessary condition for controllability.

The simplifying feature of the chain topology is that the admittance matrix is band diagonal consisting of the main diagonal and the first super- and sub-diagonals. For simplicity, we assume the conductance $G_{jk}$ vanishes for all links (which amounts to assuming losslessness) and that the susceptance is the same for all links $B_{jk} = -\frac{1}{X}$ which implies $B_{jj} = n_j \frac{1}{X}$, where $n_j$ is the number of nodes connected to node $j$ [7]. It is easy to see that the controllability results are independent of these assumptions.

Under these assumptions, the system can be written

$$\begin{align*}
\dot{x}_1 &= P_1 + \frac{1}{X} C_{12} \\
\dot{y}_1 &= Q_1 + \frac{1}{X} (D_{12} - D_{11}) \\
\dot{x}_2 &= P_2 + \frac{1}{X} (C_{21} + C_{23}) \\
\dot{y}_2 &= Q_2 + \frac{1}{X} (D_{21} + D_{23} - 2D_{22}) \\
&\vdots \\
\dot{x}_j &= P_j + \frac{1}{X} (C_{j,j-1} + C_{j,j+1}) \\
\dot{y}_j &= Q_j + \frac{1}{X} (D_{j,j-1} + D_{j,j+1} - 2D_{jj}) \\
&\vdots \\
\dot{x}_N &= P_N + \frac{1}{X} C_{N,N-1} \\
\dot{y}_N &= Q_N + \frac{1}{X} (D_{N,N-1} - D_{NN})
\end{align*}$$

By a further rescaling of time, and defining $\hat{P}_j = XP_j$, we can eliminate the constant $X$ from the equations and we have a system of the form III.8 where

$$f = \begin{pmatrix}
\hat{P}_1 + C_{12} \\
\hat{Q}_1 + D_{12} - D_{11} \\
P_2 + C_{21} + C_{23} \\
\hat{Q}_2 + D_{21} + D_{23} - 2D_{22} \\
&\vdots \\
\hat{P}_j + C_{j,j-1} + C_{j,j+1} \\
\hat{Q}_j + D_{j,j-1} + D_{j,j+1} - 2D_{jj} \\
&\vdots \\
\hat{P}_N + C_{N,N-1} \\
\hat{Q}_j + D_{N,N-1} - D_{NN}
\end{pmatrix}$$

and where $g_1$ and $g_2$ are defined as above.

We proceed to compute the node accessibility algebra. Specifically, we prove:

Theorem 4.1: The affine control system corresponding to the chain power network with inputs given by real and reactive power increments of the first node is accessible.

Proof: The starting point is the constant distribution spanned by $g_1$ and $g_2$, which we denote by $\Delta_0$. This distribution is clearly involutive, but is not drift invariant, so we first compute the Lie brackets of the drift with the control fields. To do so, we first compute the jacobian of $f$. The band diagonal structure of the admittance matrix induces a similar block band diagonal structure of the jacobian. Using the notation

$$\{J_{rs}\}_{r,s=1}^N$$

to denote the $2 \times 2$ submatrix with rows in subspace of the $r^{th}$ bus and columns in the subspace of the $s^{th}$ bus, it is easy to see from IV.10 that

![Figure 1. Chain network](image-url)
\[ J_{r,r+1} = \begin{pmatrix} -y_r & x_r \\ x_r & y_r \end{pmatrix} \overset{\text{def}}{=} (x_r^\perp, x_r) \quad r = 1, 2, \ldots, N - 1, \]
\[ J_{r,r-1} = (x_r^\perp, x_r) \quad r = 2, 3, \ldots, N, \]
\[ J_{r,r} = \begin{pmatrix} y_{r-1} + y_{r+1} & -x_{r-1} - x_{r+1} \\ x_{r-1} + x_{r+1} & 4x_r - x_{r-1} + x_{r+1} - 4y_r \end{pmatrix} \quad r = 2, 3, \ldots, N - 1, \]
\[ J_{1,1} = \begin{pmatrix} y_2 & -x_2 \\ x_2 - 2x_1 & y_2 - 2y_1 \end{pmatrix}, \]
and
\[ J_{N,N} = \begin{pmatrix} y_{N-1} & -x_{N-1} \\ x_{N-1} - 2x_N & y_{N-1} - 2y_N \end{pmatrix} \]

Since the control fields are constant, the brackets of \( g_1 \) and \( g_2 \) with the drift turn out to be the first and second columns of the jacobian, or
\[
[g_1, f] = J(f)g_1 = (y_2, x_2 - 2x_1, -y_2, x_2, 0, \ldots, 0)^T
\]
\[
[g_2, f] = J(f)g_2 = (-x_2, y_2 - 2y_1, x_2, y_2, 0, \ldots, 0)^T.
\]

To simplify the subsequent analysis, we do not take these fields as generators of the accessibility algebra, but instead use \( g_1 \) and \( g_2 \) to simplify the fields by eliminating the components in the subspace of bus 1. Therefore, we take as our new distribution
\[ \Delta_1 = \Delta_0 \oplus \text{span}\{g_3, g_4\} \]
where
\[
g_3 = (0, 0, -y_2, x_2, 0, \ldots, 0)^T,
\]
\[
g_4 = (0, 0, x_2, y_2, 0, \ldots, 0)^T.
\]

or, using the block index notation
\[
(g_3)_r = x_2 \delta_{r2}
\]
\[
(g_4)_r = x_2 \delta_{r2}.
\]

The important observation here is that since \( \det J_{21} = |x_2|^2 \) these new fields span the state space for bus 2 as long as the voltage there does not vanish. So by allowing our direct controls to interact with the drift, we extend our ability to influence the system by a distance of one connection.

Now, \( \Delta_1 \) is again involutive but is not drift invariant, so we continue the procedure by computing the brackets of \( g_3 \) and \( g_4 \) with the drift. Since these new vector fields are state dependent, we must consider both terms in the bracket. For example, for \( g_3 \), we have
\[
[g_3, f] = J(f)g_3 - J(g_3)f.
\]

The second term is irrelevant for purposes of accessibility since it always lies in the subspace corresponding to bus 2, which is already accounted for. The first term consists of state dependent linear combinations of the third and fourth columns of the jacobian of the drift. Using the previously obtained vector fields to eliminate the components in the subspaces of the first two busses as before we can take as our new fields
\[
(g_5)_r = (x_2x_3 + y_2x_3^\perp)\delta_{r3}
\]
\[
(g_6)_r = (-y_2x_3 + x_2x_3^\perp)\delta_{r3}.
\]

Since \( \det((g_5)_3 (g_6)_3) = |x_2|^2|x_3|^2 \), it follows that the two vectors are independent in the subspace of bus 3 as long as the voltages at bus 3 and bus 2 are nonzero.

The pattern should now be clear. Due to the band-diagonal nature of the jacobian of the drift, each additional layer of bracket will extend the span of the accessibility distribution into the next bus subspace. Moreover, it is easy to see that
\[
\det((g_{2r-1})_r(g_{2r})_r) = \Pi_{r=1}^{N} |x_s|^2
\]
so that after \( N \) brackets the accessibility distribution will have full rank.

We briefly discuss the situation for more general classes of networks. We choose an arbitrary node to be our control node, and call it node 1. The basic control vectors will be \( g_1 \) and \( g_2 \) as before. This distribution will be involutive as before, but will not be drift invariant unless the node is not connected to any others. Assuming it is connected, the new fields obtained by taking brackets with the drift will have nonzero components in the subspace of every bus connected to bus 1. It is easy to see that the rank increase of the accessibility algebra due to this procedure is two. One difference at this point is that the distribution constructed by adding these new fields to \( \Delta_0 \) is in general not involutive. This implies that the tower of distributions that lead to the final accessibility algebra is not unique, though the final result must be [9].

Continuing by taking brackets of these new fields with the drift, the next level will extend into all subspaces of busses two connections away from bus 1, and so forth. It’s clear from this procedure that for a connected network the vector fields generated will eventually extend to have nonzero components in the subspaces of all busses. This will happen in at most \( N \) steps, and strictly less than \( N \) for networks that are not chains. After this, additional levels of brackets will produce new vector fields that will be independent of those previously obtained. These arguments can be formalized to prove:

**Theorem 4.2:** The accessibility algebra of any node of a connected network has full rank.

**V. CONTROLLABILITY BY FEEDBACK LINEARIZATION**

Accessibility is an easily proved property that is the starting point for geometric analysis, but is not a lot of help for control synthesis in and of itself. In this section we show that the chain network is in fact controllable
via the technique of feedback linearization. The meaning of linearization in his context is not an approximation procedure, but refers to finding a coordinate change and a feedback controller such that the transformed system is linear. The main purpose of this section is to prove:

**Theorem 5.1:** The affine control system corresponding to the chain power network with inputs given by real and reactive power increments of the first node is feedback linearizable [4], [5], [9].

**Remark 5.1:** Note that for this case we have a much stronger result than input-output control. Since the entire system is feedback linearizable, the entire state can be steered to a desired set of final values.

**Proof:**

The theory of feedback linearizability is generally stated in the context of a system with outputs, though the feedback laws obtained depend on full state feedback, and not just the outputs. For the chain network with inputs given by the power increments at one end, the appropriate outputs are the states at the other end. That is, we supplement the affine control system III.8 with the outputs

\[ h_1(x) = x_N \]
\[ h_2(x) = y_N \]

Then the theory of MIMO feedback linearization (we follow the theory in [4] most closely) states that if the relative degree vector, \( (r_1, r_2) \), of the system (this will be defined below) obeys the condition that \( r_1 + r_2 = 2N \), then there exist nonlinear state feedback laws

\[ v_1 = q_P(x) + S_{P1}(x)u_P + S_{P2}(x)u_Q \]
\[ v_2 = q_Q(x) + S_{Q1}(x)u_P + S_{Q2}(x)u_Q \] (V.11)

and a local diffeomorphism \( z = T(x) \) such that the transformed system is of the form

\[ \dot{z}_1 = z_2 \]
\[ \dot{z}_2 = z_3 \]
\[ \vdots \]
\[ \dot{z}_{r_1-1} = z_{r_1} \]
\[ \dot{z}_{r_1} = v_1 \]
\[ z_{r_1+1} = z_{r_1+2} \]
\[ \vdots \]
\[ \dot{z}_{2N-1} = z_{2N} \]
\[ \dot{z}_{2N} = v_2 \]

where \( v_1 \) and \( v_2 \) are new inputs. This is the Brunovsky form of the linearized system and allows easy control synthesis to meet local controllability targets.

To prove the theorem, we need only show the relative degree result. A two input-two output system has a relative degree vector \( (r_1, r_2) \), if

\[ L_{g_j}L_{f_j}^{r_j}h_i = 0 \] (V.12)

for \( i, j = 1, 2 \) and for all \( k < r_i - 1 \), and if the \( 2 \times 2 \) matrix

\[ S(x) = \begin{pmatrix}
L_{g_1}L_{f_1}^{r_1-1}h_1(x) & L_{g_2}L_{f_1}^{r_1-1}h_1(x) \\
L_{g_1}L_{f_2}^{r_2-1}h_2(x) & L_{g_2}L_{f_2}^{r_2-1}h_2(x)
\end{pmatrix} \]

has rank two.

Since \( g_1 \) and \( g_2 \) are constant, the Lie derivative of any function with respect to either one amounts to taking the partial derivative with respect to \( x_1 \) and \( y_1 \), respectively. Thus, by definition of \( h_1 \) and \( h_2 \), V.12 is satisfied for \( k = 0 \) as long as \( N > 1 \).

For \( k \geq 1 \), from IV.10, the Lie derivatives of the outputs with respect to the drift are of the form

\[ L_{g_j}^{k}h_j = F_j^{(k)}(x_N, x_N-n, \ldots, x_{N-(k-1)}) \] (V.13)
\[ + G_j^{(k)}(x_N, x_N-n, \ldots, x_{N-(k-2)})C_{N-(k-1), N-k} \]
\[ + H_j^{(k)}(x_N, x_N-n, \ldots, x_{N-(k-2)})D_{N-(k-1), N-k} \]

for some functions \( F_j^{(k)}, G_j^{(k)} \) and \( H_j^{(k)} \) which are functions of the variables specified.

In particular, \( L_{g_j}^{k}h_j \) depends only on the states of the last \( k+1 \) busses, which means that there will be no dependence on \( x_1 \) or \( y_1 \) for \( k + 1 < N \). Therefore V.12 will hold for \( k \leq N-2 \).

To show \( S(x) \) has rank two, note that

\[ L_{g_j}^{N-1}h_j = F_j^{(N-1)}(x_N, x_N-n, \ldots, x_2) \]
\[ + G_j^{(N-1)}(x_N, x_N-n, \ldots, x_3)C_{21} \]
\[ + H_j^{(N-1)}(x_N, x_N-n, \ldots, x_3)D_{21} \]

and it follows that

\[ S = \begin{pmatrix}
-y_2G_1^{(N-1)} + x_2H_1^{(N-1)} + x_2G_2^{(N-1)} + y_2H_2^{(N-1)} \\
-y_2G_2^{(N-1)} + x_2H_1^{(N-1)} + x_2G_2^{(N-1)} + y_2H_2^{(N-1)}
\end{pmatrix} \]

and

\[ \det(S) = |x_2|^2(H_1^{(N-1)}G_2^{(N-1)} - G_1^{(N-1)}H_2^{(N-1)}) \]

Using IV.10 one can find

\[ G_j^{(k)} = -y_N-(k-2)G_j^{(k-1)} + x_N-(k-2)H_j^{(k-1)} \] (V.14)
\[ H_j^{(k)} = x_N-(k-2)G_j^{(k-1)} + y_N-(k-2)H_j^{(k-1)} \] (V.15)

from which it follows that

\[ H_j^{(k)}G_j^{(k)} - G_j^{(k)}H_j^{(k)} = |x_N-(k-2)|^2(H_1^{(k-1)}G_2^{(k-1)} - G_1^{(k-1)}H_2^{(k-1)}) \]

Since \( G_1^{(0)} = H_2^{(0)} = 1 \) and \( H_1^{(0)} = G_2^{(0)} = 0 \), we have that

\[ H_j^{(k)}G_j^{(k)} - G_j^{(k)}H_j^{(k)} = \Pi_j^{k-1}|x_N-(j-1)|^2 \]

from which it follows that

\[ \det(S) = \Pi_j^{N-1}|x_N-(j-1)|^2 = \Pi_j^{N-x_j}|x_j|^2 \]

which establishes the relative degree condition as long as none of the voltages vanish.

For general types of networks similar procedures can be applied to achieve input-output decoupling involving any pair of busses, one input, one output. The two components of the relative degree vector will be equal to the number
of connections between the selected busses. A feedback control law will exist such that the outputs will be governed by a linear system whose controls will be realized by the input powers. Since the degree vector will not generally obey the linearization condition \( r_1 + r_2 = 2N \), there will be unobservable and uncontrollable dynamics on some of the state space [9], [5]. This is discussed briefly below.

VI. Example: \( N=3 \)

We finish the paper by illustrating these results for a small chain network of three nodes. We will show local controllability by steering the system from equilibrium to a specified state nearby. The condition of local controllability implies that the system can be steered in any direction in state space. Controllability by feedback linearization implies more, since a linear system is controllable in any positive time.

The starting point of the example is a power flow solution; that is, a set of state values and power injections that constitute a zero of the drift. A convenient way to obtain the linearizing coordinates so that the system obtained is in Brunovsky normal form is as follows.

First transform to rectangular coordinates where the operating point of the grid is at the origin. That is if \( \mathbf{x}_0 \) are the coordinates of the equilibrium, then define \( \mathbf{x}' = \mathbf{x} - \mathbf{x}_0 \). Similarly denoting \( f'(\mathbf{x}') = f(\mathbf{x}' + \mathbf{x}_0) \), then the linearizing coordinates can be taken to be

\[
\begin{align*}
z_1 &= x_{N}' \\
z_2 &= L_{N}z_1 \\
&\vdots \\
z_N &= L_{N}z_{N-1} \\
z_{N+1} &= y_{N}' \\
z_{N+2} &= L_{N}z_{N+1} \\
&\vdots \\
z_{2N} &= L_{N}z_{2N-1}
\end{align*}
\]

The feedback control laws V.11 are given by

\[
\begin{align*}
q_P &= L_{N}z_1 \\
q_Q &= L_{N}z_{N+1} \\
S_{Pj} &= L_{g_j}L_{N-1}z_{j}, \quad j = 1, 2 \\
S_{Qj} &= L_{g_j}L_{N-1}z_{N+1}, \quad j = 1, 2.
\end{align*}
\]

For the numerical example we consider a simple system with \( N = 3 \). The control node 1 is connected to a generator, and the other busses are modeled as loads. The values for the powers and equilibrium values of the states are shown in Table 1. Voltages are shown per unit, angles in radians and with line inductance \( X = 0.1 \). The numbers in the example are for illustrative purposes only so that the individual bus variables can be clearly seen and are not intended to be physically realistic.

<table>
<thead>
<tr>
<th>Bus</th>
<th>P</th>
<th>Q</th>
<th>Voltage</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>165MW</td>
<td>91.1Mvar</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>75MW</td>
<td>25Mvar</td>
<td>0.9237</td>
<td>-0.1796</td>
</tr>
<tr>
<td>3</td>
<td>90MW</td>
<td>20Mvar</td>
<td>0.8959</td>
<td>-0.2885</td>
</tr>
</tbody>
</table>

Since the full system is feedback linearizable, the numerical examples are designed to show that the system can be moved in any direction in state space in any specified time. We arbitrarily choose the following two control goals. First we lower the voltage of bus three to 0.87 in one second while returning the remaining states to their equilibrium values. This is shown in Figures 3 and 4. The control is applied from 0.5 second to 1.5 second. Next we lower the voltage of bus 2 to 0.9 similarly. This is shown in Figures 5 and 6.

VII. Input-Output Control

In this section we briefly comment on a more general class of networks networks which are not fully feedback linearizable but in which the same techniques yield input-output control. Sample results are shown in a companion submission [1]. The results are dependent on the following assumption:

**Assumption 1:** We assume that the control and target nodes are connected by a unique path of buses that we label 1, 2, \ldots, \( r \). Moreover, any other paths connecting any
of the nodes $1, 2, \ldots, r$ with each other are of length greater than $r$.

Under this assumption computations such as those found in Section V show:

**Theorem 7.1:** The states at any bus connected to the control bus have controllability indices $(r, r)$, where $r$ is the minimum number of transmission lines that must be crossed in going from the control bus to the target bus.

**Proof:** The only difference is that the $F$, $G$ and $H$ functions will also depend on the variables of the busses that are within $r$ hops of the target bus, but this will not affect the values of the indices.

We also have

**Theorem 7.2:** The matrix $S$ is nonsingular.

**Proof:** Under the assumption, the recurrences V.14 and V.15 still hold, so the computations that follow and the rank condition follows.

The meaning of these results is that the target bus variables can be controlled as desired. Since we do not have full feedback linearizations the intermediate coordinates in the linearBrunovsky integrator constructed by the foregoing methods that can also be controlled by synthesis of the linear system do not amount to the ability to control specific bus variables, but only the complicated functions of them determined by the Lie derivatives. This is illustrated in [1].

**VIII. Conclusions**

In this paper we have begun a program for analyzing the effects that busses of a power system have on each other due to the network structure. The initial goal has been to model the system as simply as possible based on the power flow equations and show that the apparatus of differential geometric control can be applied to the problem of controlling the states of one bus by adjusting the power inputs of another. We have predominantly considered a simple chain network where these tools worked out particularly well due to the fact that the system is fully feedback linearizable. In a more general context, similar methods can be applied in the context of input-output decoupling. Preliminary results are shown in [1] and will be reported in more detail in a sequel.

**REFERENCES**

[1] Ball, S., S. Schaeffer, K. Wedeward and E. Barany, “Control of a singularly perturbed power flow network model using feedback linearization”, Submitted to 43rd IEEE Conf. on Decision and Control, Bahamas, December, 2004


